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Stratified rotating edge waves

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The dispersion relation is found for edge waves in a rotating stratified fluid over a constant sloping bottom. The dispersion relation is then extended to the case of arbitrary gentle bottom bathymetry. Superinertial trapped modes do not exist in the rigid-lid Boussinesq case. The effect of some of the approximations that have been made in this problem is discussed.

1. Introduction

Do internal edge waves exist? The existence of trapped edge waves in a homogeneous fluid over a sloping bottom has been known since the work of Stokes in 1846. Ursell (1952) showed that the Stokes edge mode was the first member of a family of trapped modes, with the number of discrete modes depending on the slope of the bottom. Dispersion relations for edge waves over arbitrary topography were then found in the case of gentle bottom slopes by Shen, Meyer & Keller (1968), Miles (1989) and Zhevandrov (1991, hereafter referred to as Z91) among others.

The effect of rotation was investigated by Ball (1967) in the case of an exponential bottom profile in the shallow-water approximation. Saint-Guily (1968) found the lowest two modes for the wedge, and found a difference between waves propagating with the coast on their right and left, as one would expect. Greenspan (1970) found the trapped modes for a fluid with constant stratification in a wedge. The Stokes mode was unaffected. Huthnance (1978) examined subinertial waves for arbitrary topography. Mysak (1980) reviewed trapped waves.

Evans (1989) used ideas of Whitham (1979) to write down the solution to a generalization of the Ursell problem, which he described as including Saint-Guily's and Greenspan's results. However there are some technical issues for frequencies in the internal gravity wave (IGW) band. Somewhat earlier, Ou (1980) had transformed the problem of trapped subinertial modes (frequencies less than the Coriolis frequency f) to Ursell's problem. The existence of trapped superinertial waves, however, remained an open question. Dale & Sherwin (1996) examined the question numerically but the superinertial modes they seemed to find appeared unphysical. In addition their numerical approach only considered a single bottom profile which had a vertical wall at the coast. Pringle & Brink (1999) returned to the problem and constructed WKB-like solutions, but required bottom friction and did not actually obtain edge waves. There have also been attempts (L. R. M. Maas 2003, personal communication) to construct trapped modes over a constant slope using the modes found by Wunsch (1969) but such attempts are problematic since the latter modes belong to the continuum. Table 1 summarizes some of the aforementioned work on edge waves. There also exists work on wave trapping by seamounts with stratification (e.g. Brink 1984).

Author	f	N	n	Topography	Notes
Kelvin	✓	0	0	Wall at shore	Propagates with coast on the right
Stokes	0	0	0	Wedge	
Ursell	0	0	all	Wedge	
Ball	✓	0	0	Exponential	Shallow water
Saint-Guily	✓	0	0,1	Wedge	
Greenspan	0	✓	all	Wedge	
Huthnance	✓	✓	all	General	Hydrostatic, Boussinesq, subinertial, numerical
Ou	✓	✓	all	Wedge	Rigid lid, hydrostatic, Boussinesq, subinertial
Evans	✓	✓	all	Wedge	Transformation holds outside IGW band
Zhevandrov	0	0	all	General	Gentle slope
Dale & Sherwin	✓	✓	—	Special	Rigid lid, Boussinesq, numerical , $H(0) \neq 0$
Pringle & Brink	✓	✓	—	Special	WKB-like, friction
This paper	✓	✓	all	General	Gentle slope

TABLE 1. Ranges of validity of parameters for selected previous studies on edge waves.

We build on previous work to give the full solution to the problem of edge modes in a rotating, stratified fluid over a bottom with constant slope or with arbitrary but gentle bottom profile. The problem is formulated in § 2 and the solution is presented in § 3. Special cases and common approximations are discussed in § 4, while the extension to general topography is given in § 5. Some conclusions are drawn in § 6.

2. Formulation

We consider incompressible flow of an inviscid fluid on an f-plane with no density diffusion. The x-axis points offshore and the z-axis points up. We take a basic state at rest with a stable background density profile $\rho_0(z)$ in hydrostatic equilibrium and expand. Disturbances are assumed to be proportional to $\exp(-\mathrm{i}\omega t)$, where $\omega > 0$ without loss of generality. The resulting linearized equations can be reduced to a single equation for pressure (e.g. McKee 1973):

$$\nabla^2 p + \rho_0 \frac{\partial}{\partial z}\left[\rho_0^{-1}\frac{\omega^2 - f^2}{\omega^2 - N^2}\frac{\partial p}{\partial z}\right] = 0. \tag{2.1}$$

The buoyancy frequency is defined by

$$N^2 \equiv -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}. \tag{2.2}$$

The linearized boundary conditions can also be expressed solely in terms of pressure. At the bottom $z = -H(x, y)$,

$$\frac{\omega^2 - f^2}{\omega^2 - N^2}\frac{\partial p}{\partial z} + \nabla H \cdot \nabla p + \frac{\mathrm{i}f}{\omega}\boldsymbol{k} \cdot \nabla H \times \nabla p = 0. \tag{2.3}$$

The free-surface boundary condition at $z = 0$ is

$$\frac{\partial p}{\partial z} = \frac{\omega^2 - N^2}{g} p; \tag{2.4}$$

for a rigid lid the right-hand side of (2.4) becomes zero.

A number of approximate sets of equations may be derived from the above. The hydrostatic limit corresponds to replacing $\omega^2 - N^2$ by $-N^2$ everywhere. The Boussinesq approximation replaces (2.1) by

$$\nabla^2 p + \frac{\partial}{\partial z}\left[\frac{\omega^2 - f^2}{\omega^2 - N^2}\frac{\partial p}{\partial z}\right] = 0, \tag{2.5}$$

where (2.2) becomes $N^2 \equiv -(g/\rho_*)\,\mathrm{d}\rho_0/\mathrm{d}z$, with ρ_* a reference density.

We now specialize to the case of constant N^2 and constant bottom slope angle δ. Then the basic-state density is given by $\rho_0(z) = \exp(-bz)$ with $N^2 = gb$. We take $0 < f < N$ when these frequencies are non-zero. In addition we assume an alongshore dependence $\exp(\mathrm{i}ly)$. With a right-handed coordinate system, waves with $l > 0$ propagate with the coast on their left (since $\omega > 0$). The resulting equation set is

$$\frac{\omega^2 - f^2}{\omega^2 - N^2}[p_{zz} + bp_z] + p_{xx} - l^2 p = 0, \tag{2.6a}$$

$$p_z - \frac{\omega^2 - N^2}{g} p = 0 \qquad \text{on } z = 0, \tag{2.6b}$$

$$\frac{\omega^2 - f^2}{\omega^2 - N^2} p_z + \tan\delta p_x - \frac{fl}{\omega}\tan\delta p = 0 \qquad \text{on } z = -x\tan\delta. \tag{2.6c}$$

3. Edge waves

3.1. *Stokes' edge wave*

We start by finding the lowest trapped mode, which is really just Stokes' mode, by writing $p \sim \exp(-rx - qz)$. Then we obtain three coupled equations for r, q and ω:

$$q = -\frac{\omega^2 - N^2}{g}, \quad -q\frac{\omega^2 - f^2}{\omega^2 - N^2} - r\tan\delta - \frac{fl}{\omega}\tan\delta = 0, \quad \frac{\omega^2 - f^2}{\omega^2 - N^2}(q^2 - bq) + r^2 - l^2 = 0. \tag{3.1}$$

Solving these equations gives the same result as Saint-Guily: there are two modes whose frequencies satisfy

$$\frac{\omega^2}{g}\left(1 - \frac{f}{\omega}\cos\delta\right) - l\sin\delta = 0, \qquad \frac{\omega^2}{g}\left(1 + \frac{f}{\omega}\cos\delta\right) + l\sin\delta = 0. \tag{3.2}$$

This confirms Greenspan's result that the Stokes mode is independent of stratification. It does depend on rotation. In particular, for the first mode in (3.2) to be trapped (i.e. $r > 0$), its frequency must be above the critical value $\omega_c = f/\cos\delta$.

3.2. *General solution*

The results of Greenspan (1970) and of Saint-Guily (1968) suggest that all trapped modes will take the form of a sum of exponentials. Greenspan gives the result for arbitrary stratification, but the equations are unwieldy. Evans (1989) gives a general result but does not explain what happens when the system is hyperbolic, in which case internal gravity waves can propagate (this would give superinertial trapped modes in oceanographic parlance). However, the form of (3.2) shows that this restriction must be artificial and in fact Evans' results may be extended merely by using analytic continuation.

We first reduce our system to that considered by Evans by stretching the z-coordinate using

$$z = \left(\frac{\omega^2 - f^2}{\omega^2 - N^2}\right)^{1/2}\tilde{z} = s\tilde{z} \tag{3.3}$$

so that the bottom boundary is now at $\tilde{z} = -sx \tan\delta = -x \tan\beta$. Changing dependent variable using $p = S\mathrm{e}^{-bz/2}$ leads to

$$S_{\tilde{z}\tilde{z}} + S_{xx} - k^2 S = 0, \tag{3.4a}$$

$$S_{\tilde{z}} - \lambda S = 0 \qquad \text{at } \tilde{z} = 0, \tag{3.4b}$$

$$\cos\beta S_{\tilde{z}} + \sin\beta S_x + \alpha S = 0 \qquad \text{at } \tilde{z} = -x\tan\beta. \tag{3.4c}$$

The auxiliary quantities k, λ, β and α are defined by

$$k^2 = l^2 + \frac{s^2 b^2}{4}, \quad \lambda = s\left[\frac{\omega^2 - N^2}{g} + \frac{b}{2}\right], \quad \tan\beta = s^{-1}\tan\delta, \quad \alpha = -\frac{fl}{\omega}\sin\beta - \frac{sb}{2}\cos\beta. \tag{3.5}$$

Evans' solution was derived for real s, but we disregard this and use it for all values. The dispersion relation is written

$$\lambda = (k^2 - \alpha^2)^{1/2}\sin(2n+1)\beta - \alpha\cos(2n+1)\beta \tag{3.6a}$$

$$= k\sin[(2n+1)\beta - \chi] \quad \text{where} \quad \alpha = k\sin\chi. \tag{3.6b}$$

This may be recast as a quadratic equation for l given ω. The condition for trapping can be obtained by expressing r, the offshore decay rate, for the Stokes mode from (3.1) in terms of α, λ, and β. This gives $r = (\alpha + \lambda\cos\beta)/\sin\beta$. The dispersion relation for mode n is controlled by terms in $(2n+1)\beta$, so the generalization of the trapping condition to arbitrary n is

$$\frac{\alpha + \lambda\cos(2n+1)\beta}{\sin(2n+1)\beta} = k\cos[(2n+1)\beta - \chi] > 0. \tag{3.7}$$

The second form can be deduced from (3.6) or from Evans' equation (3.21). Evans' equation also shows that the solution takes the form of a sum of exponentials of the form $\exp\{-kx\cos[(2n+1)\beta - \chi] \pm kz\sin[(2n+1)\beta - \chi]\}$, all of which ultimately decay away from the shore. However, the solutions can have zero-crossings as functions of x and of z. They are not the sum of normal modes as would be found for a flat bottom.

4. Cases and approximations

4.1. *No rotation*

This is the case considered by Greenspan (1970). With $f = 0$, the buoyancy frequency is the natural timescale, and we plot ω/N against gl/N^2. Typical results are shown in figure 1(a, b). The gravest mode is the Stokes mode. The small-ω behaviour is shown in figure 1(c). Higher modes always exist, but for each value of δ, the dispersion relation takes one of three forms. For $0 < \delta < \pi/(2n+1)$, the relation is very similar to the Stokes mode. For $\pi/(2n+1) < \delta < \pi/(n+\frac{1}{2})$, there is a high-frequency cutoff ω_h visible in figure 1(a) for $n = 2$. For $\pi/(n+\frac{1}{2}) < \delta < \pi/2$, there are two branches: one coming into the origin with an upper cutoff ω_l, and one for frequencies in a second range $\omega_{h_1} < \omega < \omega_{h_2}$ visible in figure 1(b) for $n = 2$. For δ close to $\pi/2$, the cutoffs all tend to N. Figure 1(d) shows these cutoffs as a function of δ for the case $n = 2$.

4.2. *No stratification*

This is the case considered by Saint-Guily (1968). The situation is now completely different because the y-symmetry ($l \leftrightarrow -l$) has been broken. In particular the gravest

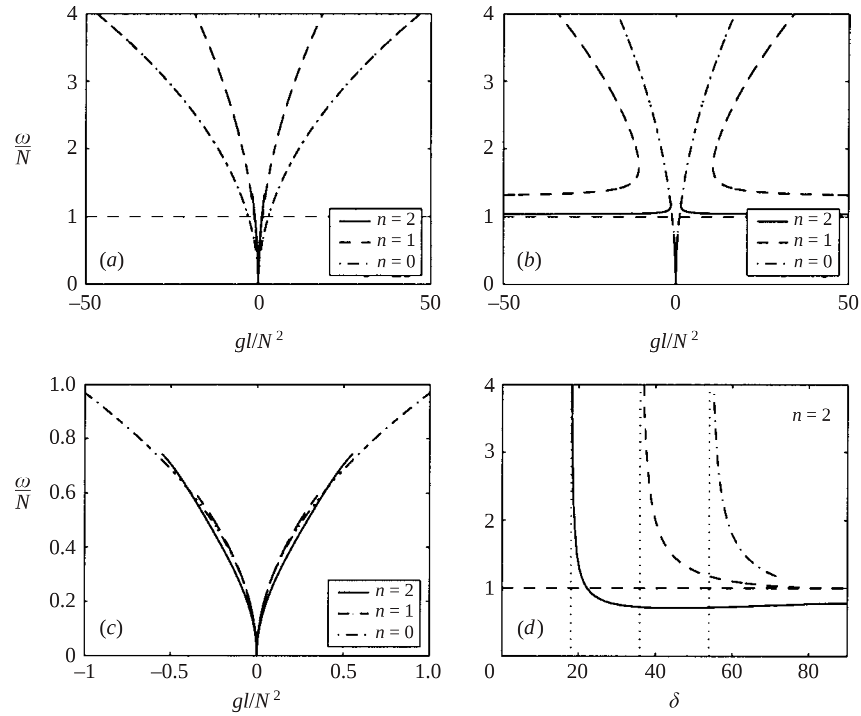

FIGURE 1. Dispersion relation for $f = 0$. (a) Bottom slope δ of 20°. (b) Bottom slope δ of 70°. (c) Enlargement of (b). (d) Existence regions for $n = 2$. To the left of the critical angle 18°, there is a solution for all ω. Above 18°, there is a solution for ω/N below the solid curve. Above 36° there is a solution for ω/N above the dashed line and below the dash-dot line (the two curves asymptote to $\omega/N = 1$ as $\delta \to 90°$).

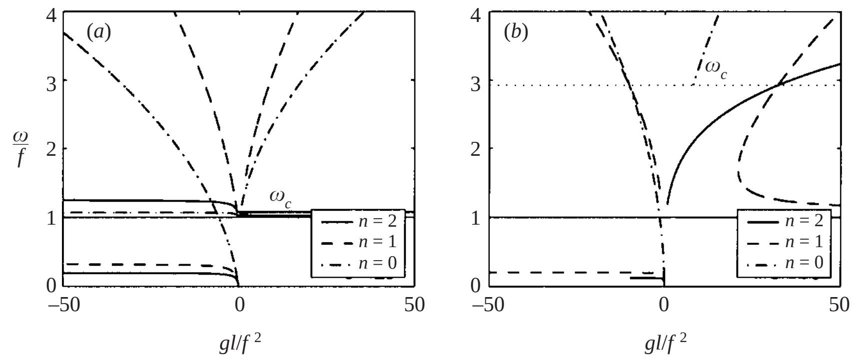

FIGURE 2. Dispersion relation for $N = 0$. The dotted lines correspond to $\omega_c = f/\cos\delta$. (a) Bottom slope of 20° ($\omega_c/f = 1.0642$). (b) Bottom slope of 70° ($\omega_c/f = 2.9238$).

mode which propagates with the coast on its left has the low-frequency cutoff ω_c. Figure 2 shows typical dispersion relations, and makes clear when the modes are actually trapped (this is hard to see in Saint-Guily's diagrams). All modes have a branch with a low-frequency cutoff. The even modes have a cutoff for branches that propagate with the coast on their left ($l > 0$, anti-Kelvin) while odd modes have a cutoff for the branch that propagates with the coast on the right ($l < 0$, Kelvin). The

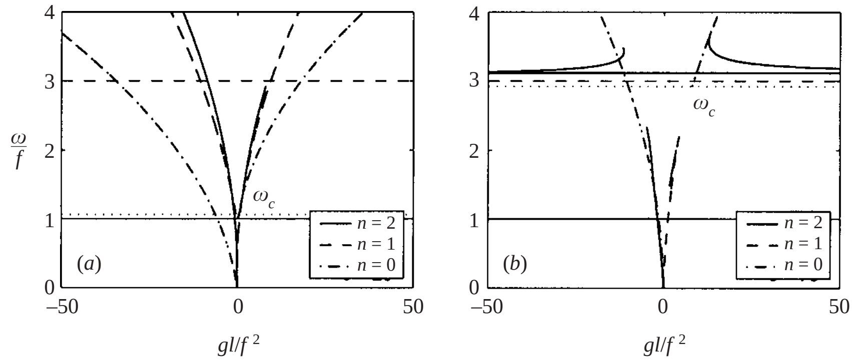

FIGURE 3. Dispersion relation for $N = 3f$. (*a*) Bottom slope of 20°. (*b*) Bottom slope of 70°. The dashed line corresponds to $\omega = N$. The thick solid horizontal curve in (*b*) is an artifact.

higher modes all have a very low-frequency branch with the coast on its left. This is a bottom-trapped topographic Rossby wave (Rhines 1970). The counterparts of these branches for Ball's (1967) topography are plotted in Munk, Snodgrass & Wimbush (1970).

4.3. *General case*

Figure 3 shows the dispersion relation in the case of the full system with $N = 3f$ for modes 0, 1 and 2. The cutoff at ω_c persists. Stratification appears to remove the low-frequency topographic Rossby waves. It also has an effect for large bottom slopes for frequencies close to N.

4.4. *Approximations*

The Boussinesq and hydrostatic approximations change (3.2) so that we can no longer say that stratification does not affect the Stokes wave. However, the effect of the hydrostatic approximation is small. Figure 4 shows one case of the dispersion relation for the lowest mode for the full equations as well as various approximations. (The rigid-lid case is discussed in the next paragraph.) The Boussinesq approximation has the very undesirable property of removing the low-frequency cutoff for the gravest mode: the two dashed lines merge just left of the l-axis and slightly below ω_c. The modes continue as a single curve down to the origin because the quadratic equation defining l acquires complex roots. These complex-conjugate roots decay offshore, so they are acceptable trapped modes. However, they are clearly a poor substitute for the full solution for small ω. It is no surprise that the Boussinesq approximation leads to problems since the wedge becomes infinitely deep far out to sea, so the assumption that the buoyancy scale is larger than the depth is clearly wrong there.

Ou (1980) considered the subinertial case with a rigid lid and the Boussinesq and hydrostatic approximations, and mapped the resulting system onto Ursell's problem. In the language of (3.4) and (3.6), $\alpha^2 = k^2 \sin^2 (2n+1)\beta$. The wavenumber l then drops out of the problem and the dispersion relation becomes

$$\frac{\omega^2}{f^2} = \frac{\sin^2 \beta}{\sin^2 (2n+1)\beta} \tag{4.1}$$

which is (4.5) of Ou (1980). The modes require $n > 0$. The trapping condition becomes $\omega l < 0$, which shows that these modes are Kelvin-like (propagating with the coast on their right). This is a highly degenerate case since the only relevant parameter is the

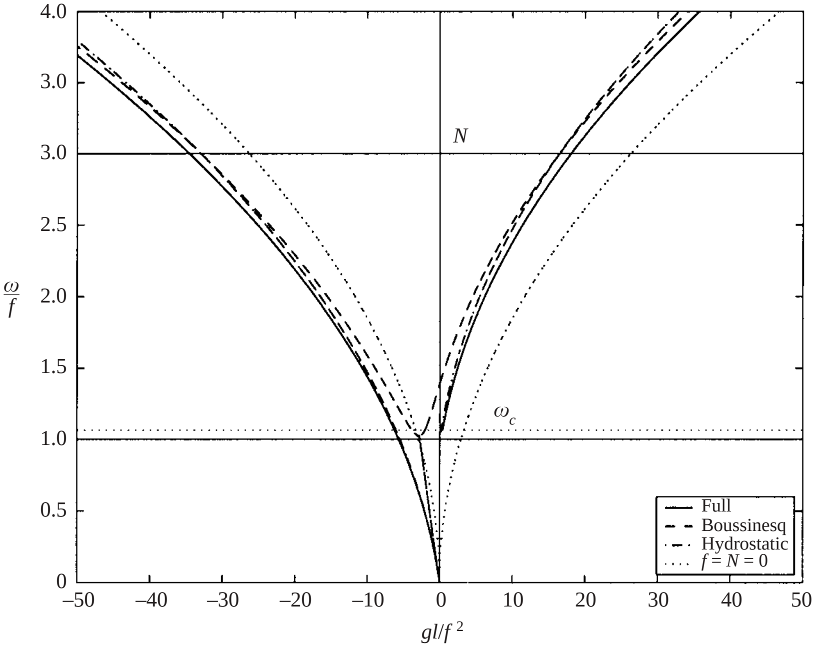

FIGURE 4. Dispersion relation for the gravest mode with $\delta = 20°$, exact and approximate.

Burger number $S = (N/f)\tan\delta$; this reduction only occurs when both the rigid-lid and Boussinesq approximations hold at the same time.

Without the hydrostatic approximation, the Ou modes exist and are still independent of l, but no longer depend on S alone: there is now separate dependence on δ and on N/f. When the Boussinesq approximation is no longer made, the l-dependence returns and we are back to the general situation above.

Superinertial modes of this kind do not exist. They would require $s^2 < 0$ and hence β imaginary, but then the right-hand side of (4.1) is less than 1 in magnitude so the mode is not superinertial, which is a contradiction. This argument does not depend on the hydrostatic approximation.

5. General topography

Z91 shows that the results of Shen *et al.* (1968) provide the correct leading-order solution to the edge-wave dispersion relation for a homogeneous fluid in the small-slope limit. We could hence follow Keller & Mow (1968, § 5) which gives the appropriate method for the stratified case (the addition of rotation does not alter the dispersion relation to leading order). However Z91 provides a neat way of writing down the result which uses a transformation similar to that of § 3.

We rewrite the set of equations (3.4) for a general bottom profile $z = -H(x)$; the only change is the bottom boundary condition which becomes

$$S_{\tilde{z}} + s^{-1}H_x S_x + \left[-\frac{fl}{\omega}s^{-1}H_x - \frac{sb}{2}\right] S = 0 \quad \text{on } \tilde{z} = -s^{-1}H(x). \qquad (5.1)$$

We now stretch variables using $\zeta = k\tilde{z}$ and $\eta = \epsilon kx$, where ϵ is the small slope near the origin, so that $kH(x) = h(\epsilon kx)$. This gives

$$S_{\zeta\zeta} + \epsilon^2 S_{\eta\eta} - S = 0, \tag{5.2a}$$

$$S_\zeta - \nu S = 0 \qquad \text{at } \zeta = 0, \tag{5.2b}$$

$$S_\zeta - \sigma S - \epsilon\gamma s^{-1} h' S + \epsilon^2 s^{-1} h' S_\eta = 0 \qquad \text{at } \zeta = -s^{-1}h(\eta), \tag{5.2c}$$

where now $\nu = \lambda/k$, $\sigma = sb/2k$ and $\gamma = fl/\omega k$ ($0 \leqslant \sigma < 1$ for non-zero l). These definitions are not the same as in Z91.†

While Z91 claims to give results valid for general stratification and rotation, this is not strictly true because of the presence of the σ term. We now modify the results of Z91 for the system (5.2). The essential difference comes from the presence of the σS term in the lower boundary condition. The leading-order dispersion relation is given by the solution of the equations

$$\kappa \tanh s^{-1}\kappa h(q) = \frac{\nu_0 - \sigma}{1 - \nu_0\sigma/\kappa^2}, \qquad \kappa = \sqrt{p^2+1},$$
$$\frac{1}{\pi\epsilon}\int_{-\infty}^{\infty} q(p, \nu_0)\,\mathrm{d}p = 2n + 1 + \frac{\gamma}{\sqrt{1-\sigma^2}}. \tag{5.3}$$

In these and following expressions, ν_0 and λ_0 are the leading-order approximations to ν and λ. The first equation in (5.3) comes from solving the normal-mode problem in ζ with $\epsilon = 0$ in (5.2); this is a problem in ζ only and the offshore coordinate η then only enters via the dependence of h on q. The result is a transcendental relation between κ and h, which differs from Z91's result for non-zero σ. The sign of the last term differs from that of Z91 because of Z91's choice of axes and of time-dependence. Its form is most easily found by following the approach of Sun & Shen (1994). This entails solving a first-order ordinary differential equation for a function $a(p)$, which leads to a phase factor whose argument gives the fraction.

We are hence led to consider the integral

$$F(\nu_0, \sigma) = \frac{2}{\pi}\int_1^{\infty} h^{-1}\left(s\kappa^{-1}\tanh^{-1}\frac{\nu_0 - \sigma}{\kappa - \nu_0\sigma\kappa^{-1}}\right)\frac{\kappa\,\mathrm{d}\kappa}{\sqrt{\kappa^2-1}}. \tag{5.4}$$

We can produce the general answer merely by using the function F so that

$$\epsilon^{-1}F(\lambda/k, \sigma) = 2n + 1 + \frac{\gamma}{\sqrt{1-\sigma^2}}. \tag{5.5}$$

This relation automatically incorporates the trapping condition unlike the quadratic used for the wedge, which needed (3.7) in addition.

For the wedge where $h = \eta$, the integral can be evaluated in closed form: $F = s(\sin^{-1}\nu_0 - \sin^{-1}\sigma)$ and the dispersion relation becomes

$$\lambda_0 = k\sin\left[s^{-1}\epsilon\left(2n + 1 + \frac{\gamma}{\sqrt{1-\sigma^2}}\right) + \sin^{-1}\sigma\right], \tag{5.6}$$

which is consistent to $O(\epsilon^2)$ with (3.6) since $s^{-1}\epsilon = s^{-1}\tan\delta = \tan\beta = \beta + O(\beta^3)$, and

$$\chi = \sin^{-1}\frac{\alpha}{k} = \sin^{-1}\left[-\frac{sb}{2k}\cos\beta - \frac{fl}{\omega k}\sin\beta\right] \approx -\sin^{-1}\sigma - \frac{\gamma\beta}{\sqrt{1-\sigma^2}} + O(\beta^2). \tag{5.7}$$

† Note that the second term of (4.1) in Z91 should be $\pm\epsilon\mu\gamma h'\phi$.

All angles are small here so $O(\epsilon^2)$ and $O(\beta^2)$ have the same effect, although if s is small, ϵ has to be very small indeed for expansions in β to make sense.

Plots of the dispersion relation are qualitatively the same as those shown in figures 1–3. The rigid-lid Boussinesq case gives $\nu_0 = \sigma = 0$ and $\gamma = \operatorname{sgn} l (f/\omega)$ in (5.4), so (5.5) becomes

$$\frac{\omega}{f} = -\frac{\operatorname{sgn} l}{2n+1}, \tag{5.8}$$

which is the small-β limit of (4.1). Subinertial modes exist, but not superinertial modes, since the right-hand side is always less than 1 in magnitude. The hydrostatic approximation is not important here, since β drops out of the dispersion relation.

6. Conclusion

The entire discrete spectrum of trapped edge waves has been found for a stratified rotating fluid over a constant slope, and the asymptotic form of the dispersion relation has been found over a gently sloping bottom. Superinertial trapped modes do not exist in the rigid-lid Boussinesq case.

As in the homogeneous case, a continuum also exists for the wedge and may be found by adapting the results of Whitham (1979) and Peters (1952) to the present situation. The condition for the existence of the continuum is $k < \lambda$, which may be solved to give $g^2 l^2 < \omega^2(\omega^2 - f^2)$.

Detailed observations of these waves do not yet appear to exist. Presumably existing observations of sea-surface displacement for edge waves (see e.g. references in Munk *et al.* 1970) incorporate some baroclinic contribution. One might also expect that they would be important in determining the behaviour of the internal tide as it approaches the shore.

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